A differential-difference approach for the thermal boundary layer under laminar conditions

Antonio Campo and Carlos Schuler
Depto. de Termodinámica, Universidad Simón Bolívar, Caracas, Venezuela

The transversal derivatives of the conservation equations for laminar boundary layer flow accounting for heat transfer are discretized by control volumes. The resulting system of ordinary differential equations of first order is solved numerically by a Runge-Kutta integration process. Based on the qualitative results of both velocity and temperature, this hybrid solution method was found to be competitive with the highly refined solution methods that represent the current state-of-the-art.

Keywords: thermal boundary layer; method of lines; control volumes

Introduction

Heat transfer by forced convection in laminar boundary-layer flows has been analyzed extensively for a thin flat plate with a sharp leading edge exposed to a variety of surface boundary conditions. Typical studies can be found in the classical book by Schlichting[1] on this subject.

With the advent of computers, it is possible to routinely solve many boundary-layer problems by approximate, purely numerical procedures. In general, the discretization procedures for the finite-difference formulations are based on Taylor's series or control volume approaches. To our knowledge, no study exists on thermal boundary layers utilizing a hybrid formulation combining finite-difference and purely differential components in the set of governing equations. This unique approach has motivated the present investigation.

The conservation equations for the boundary layers along a flat plate are first transformed into a system of ordinary differential equations of first order by discretizing the transversal derivatives via the control volume approach, while keeping the axial derivatives in continuous form. The resulting system of differential equations is then solved simultaneously by a Runge-Kutta integration scheme to determine the unknown quantities of velocity and temperature inside the boundary layers.

Numerical results using a coarse grid consisting of only seven lines at the trailing edge of the plate seem to be satisfactory for fluids covering the spectrum of Prandtl numbers between 0.01 and 15. Comparisons of both velocity and temperature profiles have been made with the classical solutions reported in the open literature by Schlichting.[1] These preliminary calculations are encouraging and are being extended to other situations involving more complex hydrodynamic and thermal conditions in boundary-layer problems.

Formulation

Consider a semi-infinite flat plate maintained at a uniform temperature t_w and placed in a fluid flow having a constant velocity u_∞ and a constant temperature t_∞. The properties of the fluid are assumed to be invariant with temperature. In addition, the viscous dissipation and axial conduction effects are neglected. Under these basic assumptions, the conservation equations for the steady, two-dimensional laminar boundary-layer flow under consideration can be written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\nu\frac{\partial^2 u}{\partial y^2} \qquad (2)$$

$$u\frac{\partial t}{\partial x}+v\frac{\partial t}{\partial y}=\alpha\frac{\partial^2 t}{\partial y^2} \qquad (3)$$

along with the imposed boundary conditions

$$x=0, \quad y>0, \quad u=u_\infty, \quad t=t_\infty \qquad (4)$$

$$x>0, \quad y\to\infty, \quad u=u_\infty, \quad t=t_\infty \qquad (5)$$

$$y=0, \quad u=0, \quad v=0, \quad t=t_w \qquad (6)$$

The leading edge of the plate is located at $x=0$, and y is measured normal to the surface of the plate.

Numerical technique

The governing Equations 1–3 are of the form

$$\frac{\partial}{\partial x}(u\phi)+\frac{\partial}{\partial y}(v\phi)=\frac{\partial}{\partial y}\left(\Gamma_\phi\frac{\partial\phi}{\partial y}\right)+S_\phi \qquad (7)$$

where ϕ denotes a general dependent variable: 1, u, and t for conservation of mass, momentum, and energy, respectively.

In this section, we outline the mathematical and numerical concepts underlying the finite-difference formulation of Equation 6. Correspondingly, a technique that replaces a partial differential equation in two independent variables by an appropriate system of ordinary differential equations in one of these variables will be explored here. This procedure is commonly identified as the method of lines and is described by Liskovets.[2] Accordingly, if the independent variables of a partial differential equation are x and y, as in Equation 7, the region of integrations is divided into strips parallel to the flat plate by lines $y=$ constant (Figure 1). Then, the partial derivatives in the y direction are discretized by finite-difference expressions. Therefore, this combined procedure leads to a system of first-order ordinary differential equations for ϕ along each line in terms of the only independent variable, x.

Address reprint requests to Dr. Campo at the Mechanical Engineering Department, School of Engineering, Florida International University, University Park, Miami, FL 33199, USA.

Received 3 July 1986; accepted for publication 1 December 1986

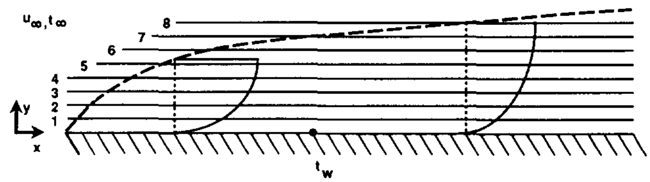

Figure 1 Distribution of lines

Following Patankar,[3] the discrete differences in Equation 7 will be based on control volume balanced in the y direction only. In view of this, a physical region is divided into a set of control volumes, such as those shown by dashed lines in Figure 2. These control volumes are of size Δx (infinitesimal) by Δy (finite) and are shown centered on line P. The faces lie at n and s, midway between lines P and N and P and S, respectively. When applied to the control volume P, the integral balance of Equation 7 yields

$$\int_s^n \frac{\partial}{\partial x}(u\phi)\,dy+(v\phi)_n-(v\phi)_s$$
$$=\Gamma_\phi\left(\frac{\partial \phi}{\partial y}\right)_n-\Gamma_\phi\left(\frac{\partial \phi}{\partial y}\right)_s+\int_s^n S_\phi\,dy \tag{8}$$

Assuming a unit depth in the z direction, the integrals are evaluated over the areas $\Delta y\times(1)$ (Equation 1) of the faces of the control volume. To proceed with the analysis, the integrands in Equation 8 are approximated in a way that the quantities u, ϕ, and S_ϕ are assumed constant between the boundaries n and s of the control volume. Therefore, the end result is given by a general ordinary differential equation of first order, which may be written as

$$\frac{d}{dx}(u_P\phi_P)=\frac{\Gamma_\phi}{\Delta y}\frac{\partial \phi}{\partial y}-\frac{1}{\Delta y}(v\phi)_s^n+S_{\phi P} \tag{9}$$

This single equation describes the continuous variation of each of the transported quantities along the lines drawn in the x direction parallel to the plate.

In the light of the foregoing discussion, and for simplicity using a uniform spacing in the y direction, the set of descriptive Equations 1–3 is transformed to

$$\frac{du_P}{dx}=-\frac{1}{\Delta y}(v_n-v_s) \qquad \text{mass} \tag{10}$$

$$\frac{d}{dx}(u_P^2)=\frac{v}{\Delta y}\left(\frac{\partial u}{\partial y}\right)_s^n+\frac{1}{\Delta y}(v_s u_s-v_n u_n) \qquad \text{momentum} \tag{11}$$

$$\frac{d}{dx}(u_P t_P)=\frac{\alpha}{\Delta y}\left(\frac{\partial t}{\partial y}\right)_s^n-\frac{1}{\Delta y}(vt)_s^n \qquad \text{energy} \tag{12}$$

Moreover, these equations may be rearranged as follows:

$$v_n=v_s-(\Delta y)\frac{du_P}{dx} \tag{13}$$

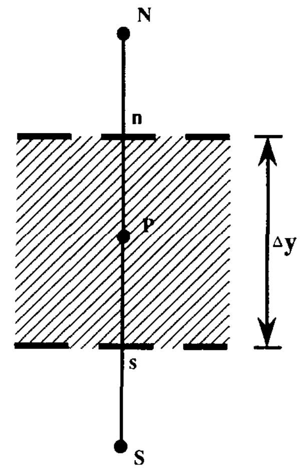

Figure 2 Control volume in one dimension

$$\frac{du_P}{dx}=\frac{1}{\Delta y(2u_P-u_n)}\left[v\left(\frac{\partial u}{\partial y}\right)_s^n-v_s(u_s-u_n)\right] \tag{14}$$

$$\frac{dt_P}{dx}=\frac{1}{u_P\Delta y}\left(\alpha\frac{\partial t}{\partial y}-vt\right)_s^n-\frac{t_P}{u_P}\frac{du_P}{dx} \tag{15}$$

The system of first-order ordinary differential equations given by Equations 14 and 15 provides the axial velocity and temperature along each line of the control volume subject to the initial conditions

$$x=0, \qquad u_P=u_\infty, \qquad t_P=t_\infty \tag{16}$$

In addition to this, the transversal velocity may be calculated directly from the evaluation of Equation 13.

The necessary number of lines drawn inside the boundary layer is determined for each x step by adding four new lines to the first 4 originally selected (Figure 1) so that the conditions

$$\frac{u}{u_\infty}>0.99 \qquad \frac{t-t_w}{t_\infty-t_w}>0.99 \tag{17}$$

are automatically satisfied. Thus the required number of lines increases in the downstream direction and follows the natural growth of the boundary layers, as Figure 1 shows.

Results and discussion

Numerical solutions of the descriptive equations (Equations 13–15) have been carried out for $\mathrm{Pr}=0.01$, 0.7, and 15 using a fourth-order Runge-Kutta subroutine. With a step size of 10^{-6}, Figure 3 shows the resulting values for the dimensionless axial velocity as a function of η using seven lines uniformly

Notation

Pr	Prandtl number, ν/α
Re_x	Local Reynolds number, $u_\infty x/\nu$
S_ϕ	Source term, Equation 1
t	Temperature
t_w	Wall temperature
t_∞	Free stream temperature
u	Axial velocity component
u_∞	Free stream velocity
$\bar{u}$	Nondimensional value of u, $\dfrac{u}{u_\infty}$
v	Transversal velocity component
x	Axial coordinate
y	Transversal coordinate
α	Thermal diffusivity
Γ_ϕ	Diffusion coefficient, Equation 7
Δ	Difference
η	Similarity variable, $y\dfrac{u_\infty}{2\nu x}$
θ	Nondimensional temperature, $(T-T_\infty)/(T_w-T_\infty)$
ν	Kinematic viscosity
ϕ	Generalized variable, Equation 1

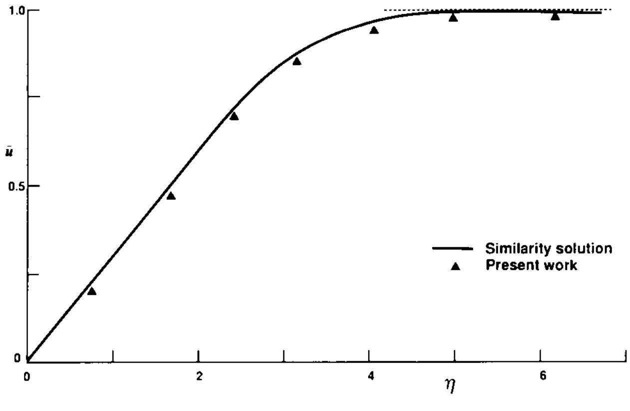

Figure 3 Laminar velocity profile

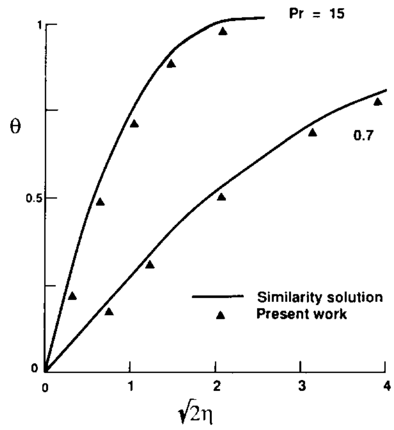

Figure 4 Laminar temperature profile for various Prandtl numbers

distributed. A comparison between these values and the predicted velocities of the similarity solution reported in Schlichting[1] appears to be reasonably good.

Attention may now be turned to the corresponding temperature profiles plotted in Figure 4 as a function of η for parametric values of the Prandtl number. The calculated temperatures using only seven lines with equal intervals are compared with those provided by the similarity solution. Agreement is quite good for both cases tested having $Pr = 0.7$ and 15.

As a final remark, it can be said that an explicit finite-difference procedure incorporating a combination of differential and finite-difference components has been developed for the analysis of laminar boundary-layer flows over flat plates. Its salient features are simplicity, directness, coarse grid, stability, and no required iterations. The predictions for velocity and temperature having an average CPU time of 25 s on a PDP-10 computer agree well with the classical solutions. From this discussion, it appears that the success of the methodology may justify further work, including additional hydrodynamic and thermal effects characterizing more complex situations in boundary-layer theory.

In addition to these cases tested, Figure 4 also shows a limiting case of $Pr = 0.01$. Further examination of this figure shows the results for $Pr = 15$ using a grid of 14 lines overlapping with those using 7 lines.

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